An Aspect of Variable Population Poverty Comparisons

Nicole Hassoun\textsuperscript{a}
S. Subramanian\textsuperscript{b}

\textsuperscript{a}Carnegie Mellon University, 
Philosophy Department, 
Pittsburgh, PA 15213, 
U.S.A. 
E-mail: nhassoun@andrew.cmu.edu

\textsuperscript{b}Madras Institute of Development Studies, 
79 Second Main Road, 
Gandhinagar, Adyar, 
Chennai – 600 020, 
Tamil Nadu, 
India. 
E-mail: subbu@mids.ac.in

\textbf{Corresponding Author:}

Nicole Hassoun, 
Carnegie Mellon University, 
Philosophy Department, 
Pittsburgh, PA 15213, 
U.S.A. 
E-mail: nhassoun@andrew.cmu.edu 
Telephone: 412.268.8946 
Fax: 412.268.1440
Abstract:

This paper demonstrates that the property of Replication Invariance, generally considered to be an innocuous requirement for the extension of fixed-population poverty comparisons to variable-population contexts, is incompatible with other plausible variable- and fixed-population axioms. This fact raises questions about what constitutes an appropriate headcount assessment of poverty, in terms of whether one should focus on the proportion, or the absolute numbers, of the population in poverty. This observation, in turn, has important implications for tracking poverty and setting targets for its reduction or elimination.

Keywords:

Variable Populations, Fixed Populations, Replication Invariance, Population Focus, Income Focus, Impossibility Result

JEL Classification:

I32, D63, D30.
1. Introduction

Poverty and its alleviation are amongst the most major concerns confronting development economists. A preliminary task in tackling the problem of poverty has to do with measuring it satisfactorily. This requirement is of the first importance for poverty assessment, target-setting, tracking, and monitoring. There is, however, an elementary difficulty that is often overlooked in quantitative assessments of poverty. The difficulty is the following one. The simplest and most direct measures of poverty relate to some headcount of those living in poverty (typically those with incomes below a threshold called the poverty line). There are at least two immediately intuitive ways of reckoning such a headcount: (a) in terms of a headcount ratio, which is the proportion of a population in poverty; and (b) in terms of an aggregate headcount, which is the absolute numbers of a population in poverty. Does it matter which headcount one employs? Should one care?

Here are a couple of examples which suggest that it does matter, and that one should care. Chakravarty, Kanbur and Mukherjee (2006) observe that the two headcount indicators just alluded to, give inconsistent diagnoses of changes in the magnitude of world poverty between 1987 and 1998. On the two dollars per day poverty line, the headcount ratio, or the proportion of people in poverty, declined from 61 per cent to 56.1 percent. The aggregate headcount, or the total number of people in poverty, rose from 2.5 billion to 2.8 billion. Without deciding which measure is best, one cannot even be sure if global poverty has declined or increased between 1987 and 1998. Or take the case considered by Subramanian (2005) of two States of the Indian Union, Orissa and Maharashtra, in the year 1987-88. The headcount ratios for Orissa and Maharashtra were 57.9 per cent and 35.8 per cent respectively, but this ranking is inverted by the aggregate headcounts for the two States, at 17.5 millions and 26.3 millions respectively. How does one attend to issues of alleviation, prioritization, and allocation of resources in a situation where even diagnosis is rendered uncertain by a plurality of plausible headcounts and disagreement amongst them?

In both of the examples just considered, one is making variable population comparisons –one is concerned with comparing poverty across populations of differing
sizes. In the first example, one is engaged in a time-series poverty comparison (a comparison of poverty in the same region at two different points of time). In the second example, one is engaged in a cross-section comparison (a comparison of poverty in two different regions at the same point in time). Variable population comparisons are the only sorts of comparison that we routinely and systematically encounter in the world as we know it. However, for analytical purposes and in order to fix our ideas with respect to the sorts of properties which we may want a poverty measure to possess, we typically consider comparisons of poverty across (hypothetical) populations of the same size: these are fixed population comparisons. Thus it is that much of the canonical work in poverty measurement relates to fixed population comparisons.

This is a simplification, but in order to permit a re-entry into the real world of variable population comparisons, one needs to find a bridge which will effect the transition from fixed population to variable population settings. This could entail – by way of example - thought experiments in which one asks what should be seen to be happening to poverty if a population were to split into two identical halves or if it should join up with another identical population of the same size. It is a standard feature of the poverty measurement literature (and indeed of the literature on the measurement of both welfare and inequality) to invoke the Replication Invariance Axiom, which allows poverty comparisons in fixed population contexts to be extended to poverty comparisons in variable population contexts. Recently, however, several scholars, motivated by philosophical work on population ethics\(^1\), have questioned whether it is as logically straightforward and ethically unexceptionable as it has been presumed to be, to effect the transition from fixed to variable population contexts via a replication invariance postulate (see Kundu and Smith, 1983; Subramanian, 2002, 2005a, 2005b; Paxton, 2003; Chakravarty, Kanbur and Mukherjee, 2006; Kanbur and Mukherjee, 2007; Hassoun, 2009). The issue, at a certain mundane level, has to do with the appeal of alternative reckonings of a headcount measure of poverty, as we shall presently see.

Replication invariance is the purportedly unexceptionable requirement that the extent of poverty in a situation should remain the same if the population is replicated any number of times. The present paper demonstrates, however, that this property is inconsistent with some other canonical axioms for poverty measurement in the presence
of the *Population Focus* axiom. Population Focus requires an addition to the non-poor population to leave the extent of poverty unchanged.

What do the headcount ratio and the aggregate headcount have to do with the Replication Invariance and Population Focus axioms? To see what is involved, consider a situation in which we have two societies, such that the income vector for Society 2 is a 2-replication of that for Society 1 (that is, there are exactly twice as many people at each income level in Society 2 as there are in Society 1). If there are $q_1$ poor people in a population of $n_1$ persons in Society 1, then since the income vector in Society 2 is a 2-replication of the vector in Society 1, the number of poor people in Society 2 is $q_2 = 2q_1$, while the size of the population in Society 2 is $n_2 = 2n_1$. Notice now that the headcount ratio for Society 1 is $q_1 / n_1$, exactly the same as for Society 2 ($q_2 / n_2 = 2q_1 / 2n_1 = q_1 / n_1$): the headcount ratio satisfies the Replication Invariance Axiom. The aggregate headcount, however, does not satisfy Replication Invariance. It says that there is twice as much poverty in Society 2 as in Society 1 (since $q_2 = 2q_1$).

Suppose now that Society 3 is derived from Society 1 by adding one non-poor person to the population. Then, in moving from Society 1 to Society 3, the aggregate headcount will remain the same at $q_1$, whereas the headcount ratio will decline from $q_1 / n_1$ to $q_1 / (n_1 + 1)$: the aggregate headcount satisfies, while the headcount ratio violates, the Population Focus Axiom.

The practical issue of poverty diagnosis with which we started out, and which involves the conflicting judgments on poverty ranking that could be delivered by the headcount ratio and the aggregate headcount, thus turns out to be related to the rather more foundational issue of the axiom systems underlying poverty measurement. The nature of the difficulty is brought out in an impossibility result which is stated, proved and discussed in this paper. The impossibility result underlines a tension between two ways of thinking about the extent of poverty – as the proportion of people in a population who are poor (a notion supported by the replication invariance axiom) and as the absolute number of poor people in a population (a notion supported by the population focus axiom). The discussion of axiomatics may appear to be situated in some distant and arcane concern with problems in the abstract realm of social choice theory. However, and
as we have been at some pains to point out in this introductory section, the seemingly rarefied problem considered in this paper is actually linked to very pragmatic issues relating to the assessment and alleviation of poverty.

In general, when some headcount of the poor is incorporated as a component of a poverty measure, it becomes a matter of importance whether the headcount is a fraction or a whole number. The present paper flags the significance of the tension between thinking about poverty in terms of proportions and absolute numbers for quantitative poverty assessment and, hence, for both creating and evaluating development policy aimed at addressing the problem of deprivation.

2. Concepts and Axioms of Poverty Measurement

An income distribution is an \( n \)-dimensional vector \( \mathbf{x} = (x_1, \ldots, x_i, \ldots, x_n) \), in which the typical element \( x_i \) stands for the (non-negative) income of person \( i \) in a community of \( n \) individuals, \( n \) being a positive integer. The poverty line, designated by \( z \), is a positive level of income such that anybody with an income less than \( z \) is labeled poor.\(^2\)

The set of all individuals whose incomes are represented in the distribution \( \mathbf{x} \) is \( N(\mathbf{x}) \), whose cardinality is \( n(\mathbf{x}) \). Given that the poverty line is \( z \), \( Q(\mathbf{x}; z) \) is the set of poor people in \( \mathbf{x} \), that is, \( Q(\mathbf{x}; z) = \{i \in N(\mathbf{x}) | x_i < z\} \). For any \( \mathbf{x} \), and any \( z > 0 \), the vector of poor incomes is designated by \( \mathbf{x}_z^P \), while the vector of non-poor incomes is designated by \( \mathbf{x}_z^R \). If \( X_n \) is the set of all \( n \)-dimensional income vectors, then the set of all conceivable income vectors is given by \( \mathbf{X} \equiv \bigcup_{n=1}^{\infty} X_n \).

We shall let \( \Pi \) stand for a weak binary relation of poverty defined on \( \mathbf{X} \): specifically for all \( \mathbf{x}, \mathbf{y} \in \mathbf{X} \), we shall write \( \mathbf{x} \Pi \mathbf{y} \) to signify that there is no more poverty in \( \mathbf{x} \) than in \( \mathbf{y} \). The asymmetric and symmetric factors of \( \Pi \) are represented by \( \overline{\Pi} \) and \( \Pi \) respectively:

\[
\forall \mathbf{x}, \mathbf{y} \in \mathbf{X} : \mathbf{x} \overline{\Pi} \mathbf{y} \leftrightarrow [\mathbf{x} \Pi \mathbf{y} \land \neg (\mathbf{y} \Pi \mathbf{x})] ; \text{ and }
\forall \mathbf{x}, \mathbf{y} \in \mathbf{X} : \mathbf{x} \Pi \mathbf{y} \leftrightarrow [\mathbf{x} \Pi \mathbf{y} \land \mathbf{y} \Pi \mathbf{x}] .
\]
That is, we shall write \( x \succ y \) to signify that there is less poverty in \( x \) than in \( y \), and \( x \succeq y \) to signify that there is exactly as much poverty in \( x \) as there is in \( y \). It will be assumed that \( \Pi \) is reflexive (\( \forall x \in X : x \Pi x \)) and transitive (\( \forall x, y, z \in X : x \Pi y \& y \Pi z \rightarrow x \Pi z \)).

We shall not, however, insist that \( \Pi \) be complete (which is the requirement that \( \forall x, y \in X : x \Pi y \) or \( y \Pi x \)). \( \Pi \), in other words, will be assumed to be a quasi-order.

Throughout, we shall assume that the binary relation \( \Pi \) is anonymous, that is, for any given poverty line and all \( x, y \in X \), if \( x \) is merely a permutation of \( y \), then \( x \Pi y \) - which is just another way of saying that poverty assessments are impervious to the personal identities of individuals. The anonymity axiom enables variable population (either cross-section or time-series) poverty comparisons to be performed in such a way as to suggest that, of a pair of distributions under comparison, one distribution can be seen to have been derived from the other through a population increment or decrement.

Presented below are a set of axioms for comparisons of poverty across both fixed and variable populations. Since the axioms are generally well-known, and in any event their import is reasonably clear, we shall not spend much time in explaining their meaning and significance. It will be taken as read that in everything that follows, \( z \) is a strictly positive scalar. Also, \( \Pi \) will be taken to belong to the set \( \mathcal{R} \) of all anonymous quasi-orders.

**Income-Focus.** For all \( z \), and for all \( x, y \in X \), if \( N(x) = N(y) \) and if \( y^p \) is either identical to, or a permutation of, \( x^p \), then \( x \Pi y \).

This axiom is usually just called the Focus Axiom. It requires that any change in non-poor incomes that leaves the numbers of individuals on either side of the poverty line unchanged, ought not to have any impact on the extent of poverty.

**Monotonicity.** For all \( z \), and for all \( x, y \in X \), if \( N(x) = N(y) \) and \( x_i = y_i \forall i \in N(x) \setminus \{j\} \) for some \( j \) such that \( j \in Q(y;z) \& x_j > y_j \), then \( x \succeq y \).
The Monotonicity Axiom states that, other things equal, an increase in a poor person’s income should reduce poverty.

**Transfer.** For all \( z \), and for all \( x, y \in X \), if \( N(x) = N(y) \), and
\[
x_j = y_j \forall i \in N(x) \setminus \{j, k\} \text{ where } j \text{ and } k \text{ are such that } j \in Q(y; z), k \in N \setminus Q(x; z),
\]
\[
x_j = y_j + \delta, x_k = y_k - \delta , \text{ and } 0 < \delta \leq (y_k - y_j)/2 , \text{ then } x \sqsupseteq y .
\]

The Transfer Axiom presented here is weaker than Donaldson and Weymark’s (1986) Weak Downward Transfer axiom. As we have stated it, the Transfer Axiom demands that, *ceteris paribus*, a mutually rank-preserving transfer of income from a non-poor person to a poor person that keeps the former non-poor, should reduce poverty.

**Replication Invariance.** For all \( z \), for all \( x, y \in X \), and any positive integer \( k \), if \( y = (x, \ldots, x) \) and \( n(y) = kn(x) \), then \( x \sqsupseteq y \).

Replication Invariance requires the extent of poverty to remain unchanged by any \( k \)-fold replication of a population.

**Population-Focus.** For all \( z \), all \( x \geq z \), and all \( x, y \in X \), if \( x_z^p = y_z^p \) and \( y_z^r = (x_z^r, x) \), then \( x \sqsupseteq y \).

Population-Focus, which corresponds to a property that Paxton (2003) calls the Poverty Non-Invariance Axiom, requires an addition to the non-poor population to leave the extent of poverty unchanged. Motivationally, it is very much in the spirit of the standard (Income-) Focus Axiom, which says poverty is not affected by changes in non-poor incomes which leave the income distribution amongst the poor unchanged. Population-Focus is directly opposed in spirit to an axiom of Non-Poverty Growth, due to Kundu and Smith (1983), which stipulates that poverty should decline with the addition of a non-poor person to the population.
3. An Impossibility Result

It is well-known that, as a measure of poverty, the headcount ratio (or indeed any purely headcount appraisal) has a number of short-comings. In a fixed-population context, the remedy for the deficiencies of the headcount ratio has been sought in more sophisticated measures (such as the per capita income-gap ratio and Sen’s index) which are capable of satisfying the Monotonicity and Transfer axioms. In a variable populations setting, however, the following result demonstrates that there are further problems with the headcount-ratio view of poverty.

Proposition.\(^3\) There exists no \( \Pi \in \mathcal{R} \) which satisfies Monotonicity (M), Replication Invariance (RI), and Population-Focus (PF).

Proof. Let \( z \) be the poverty line, and let \( a, b \) and \( c \) be three income distributions such that \( a = (x, x, \ldots, x) \), \( b = (x, x, \ldots, x, y) \) and \( c = (x, x, \ldots, x) \), with \( 0 \leq x < z < y \), and \( n(a) = n(b) = n(c) + 1 \). Define a fourth income distribution \( d \) such that \( d \) is an \( n(c) \)-replication of \( a \). It follows, then, that \( d \) is also an \( n(a) \)-replication of \( c \). Indeed, \( d \) is just the \( n(a)n(c) \)-dimensional vector \( (x, \ldots, x) \). By Axiom RI, \( \overrightarrow{a}d \) and \( \overrightarrow{d}c \), leading, through transitivity over the triple \( \{a, d, c\} \), to \( \overrightarrow{a}c \). Further, by Axiom M, \( \overrightarrow{b}a \). Given \( \overrightarrow{b}a \) and \( \overrightarrow{a}c \), transitivity over the triple \( \{b, a, c\} \) implies \( \overrightarrow{b}c \). But this contradicts \( \overline{\{b}c \}, as dictated by Axiom PF. \( \blacksquare \)

The following Corollary to the Proposition just proved is also true:

Corollary.\(^4\) There exists no \( \Pi \in \mathcal{R} \) which satisfies Transfer (T), Replication Invariance (RI), and Population-Focus (PF).

Proof. It will first be claimed that Population Focus and Transfer together imply Monotonicity. To see the truth of this claim, consider the following. Let \( n \) be a positive integer, and let \( x, y, u \) and \( v \) be four income distributions such that \( x = (x_1, \ldots, x_n) \); \( y = (y_1, \ldots, y_n) \) with \( y_i = x_i \forall i \neq j \) and \( j \) is a person for whom \( x_j < z \) (that is, \( j \) is poor) and \( y_j = x_j + \Delta \) (with \( \Delta > 0 \)); \( u = (x, z + \Delta) \); and \( v = (y, z) \). It is easy to see that \( y \overline{\Pi}v \)
by Population Focus; $v\bar{\Pi}u$ by Transfer; $y\bar{\Pi}u$ by transitivity over the triple $\{y, v, u\}$; $u\bar{\Pi}x$ by Population Focus; and $y\bar{\Pi}x$ by transitivity over the triple $\{y, u, x\}$: but then $y\bar{\Pi}x$ is precisely what is demanded by Monotonicity. It follows then, as claimed, that the non-existence of a $\Pi \in \mathcal{R}$ which satisfies Monotonicity, Replication Invariance and Population-Focus entails, as a corollary, the non-existence of a $\Pi \in \mathcal{R}$ which satisfies Transfer, Replication Invariance and Population-Focus.

It is relevant here to note that Sen’s (1976) seminal work, reflected in the quest for income-responsive and distribution-sensitive poverty measures, was motivated precisely by the failure of the headcount ratio to satisfy fixed-population axioms like Monotonicity and Transfer. In a variable population context, the headcount ratio is the archetypal Replication Invariance-satisfying measure. This paper has shown that, when we employ a population-focus axiom, it is impossible to simultaneously satisfy Replication Invariance and Monotonicity or Transfer. Hassoun (2009) suggests that similar results hold for specific real-valued poverty indices which incorporate the headcount ratio: she shows that a decline in poor persons’ incomes or a regressive transfer of incomes between poor persons, are nevertheless compatible with a decline in poverty on many common measures. This is so as long as these changes are accompanied by a sufficiently large decline in the headcount ratio owing to an increase in the non-poor population.

Finally, and as a referee has pointed out, there is a certain analogy between the Replication Invariance-Population Focus conflict and the Scale Invariance-Translation Invariance conflict in poverty measurement. Scale Invariance requires that a uniform scaling up or down of an income distribution and the poverty line should leave poverty unchanged. Translation Invariance requires that a uniform increment or decrement to each income in a distribution and the poverty line should leave poverty unchanged (see Zheng 1994). In a similar spirit – but with the focus of attention shifted from incomes to populations – Replication Invariance requires that a uniform scaling up or down of the population size should not affect poverty. Population Focus demands that additions or
subtractions to the non-poor population which leave the income-distribution of the poor unchanged ought not to affect poverty.

4. Concluding Observations

In much of the standard literature on poverty measurement, it has for long been held that Monotonicity, Transfer and Replication Invariance are three desirable axioms for a poverty index. This paper has argued that if some sort of population-focus axiom is also accepted, then one either has to reject Replication Invariance or each of the following: Monotonicity and Transfer. Much of the quest for ‘sophisticated poverty measures’ in the poverty literature has been motivated precisely by the appeal of the Monotonicity and Transfer axioms. These attributes have by now become an ingrained and accepted feature of poverty indices, and for reasons that appear to be sound enough to make one wonder on what grounds one can possibly reject them.

As for Population Focus, we suggest that one might be disposed favourably toward it for the same reason that what is popularly known as the Focus Axiom has been traditionally taken so seriously in the poverty measurement literature. The Focus Axiom is really an Income-Focus Axiom, which demands (in a fixed-population context) that measured poverty ought not to be sensitive to increases in non-poor incomes. The rationale for this requirement is that poverty is a feature of the poor, and not of the general population: in making poverty comparisons, one ought – in this view - to focus attention only on the income distribution of the poor. A similar rationale suggests focusing attention only on the population of the poor. That is, if poverty is a characteristic of the poor, then additions to the non-poor population – exactly like additions to non-poor incomes - ought not to affect the magnitude of poverty. Both income-focus and population-focus axioms can then be seen as being allied to Broome’s (1996) ‘Constituency Principle’ in population ethics, a principle which, in the present context, would require assessments of the extent of poverty to be based exclusively on information regarding the constituency of the poor.

On the other hand, it is not immediately obvious that there is some inherently indispensable ethical appeal attaching to Replication Invariance. And yet, rejecting Replication Invariance is not a painless option to implement. Commonly employed
procedures for poverty comparison, relying on dominance criteria such as Stochastic Dominance, may not lend themselves to meaningful interpretation if one does not subscribe to Replication Invariance. Briefly, then, it would appear that variable populations present a non-trivial problem for the formulation of satisfactory poverty indices.

The preceding discussion has been largely in the nature of a technical or analytical summary of the essay’s implications for aspects of coherence and interpretation in the measurement of poverty. As the introductory section heavily underscored, these abstract considerations bear on important practical issues with which a development economist must contend. Specifically, the paper invites attention to the fact that one should care, and it does make a difference, whether one takes a headcount ratio or aggregate headcount view of the prevalence of poverty. As Thomas Pogge (2010) has insightfully pointed out, it matters a lot that the Millennium Declaration promised to halve, by 2015, the fraction of the 2000 population in extreme poverty, whereas the earlier Rome Declaration promised to halve, by 2015, the number of undernourished people in 1996. For, if we only achieve the former goal, the world will contain 165 million more extremely poor people in relation to the latter goal. In the end, the tension between thinking about poverty as the proportion of people in a population who are poor and as the absolute number of poor people could not be more important for the tasks of poverty assessment, target-setting, tracking, and monitoring.
Acknowledgements:

The authors would like to thank John Weymark, Teddy Seidenfeld, Lucio Esposito, John Mumma, Efthymios Athanasiou, and the audience at the Social Choice and Welfare conference in Moscow, Russia, for very helpful comments. Special mention must be made of the quite exceptionally conscientious and insightful suggestions for improvement made by two anonymous referees and the Editor of this Journal.
NOTES


2. This is what Donaldson and Weymark (1986) call the ‘weak’ definition of the poor, which excludes an individual on the poverty line from the count of the poor.

3. We are grateful to both referees for help with establishing a much tidier proof than we had initially provided.

4. We thank one of the referees for showing us how this result can be established as a corollary to the Proposition: in our original proof, the Proposition was not employed in the proof of the Corollary.
References


